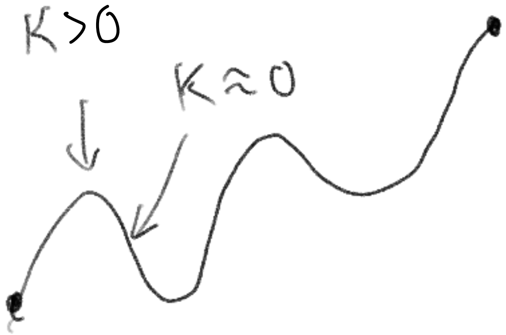
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Enrichment Programme for Young Mathematics Talents

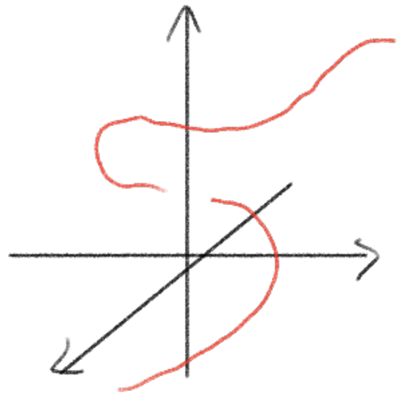
Towards Differential Geometry

CHENG Man Chuen

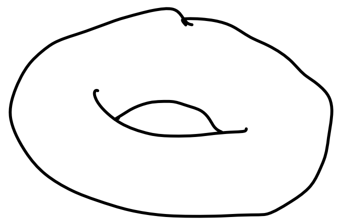
Objects to study



Plane Curve ($\mathbb{R}^2$)

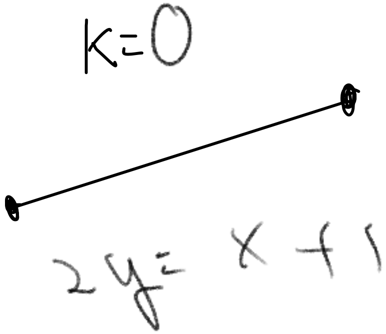


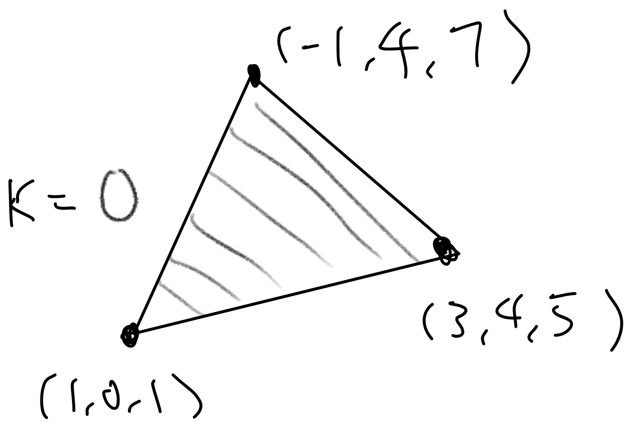
Space Curve ($\mathbb{R}^3$)

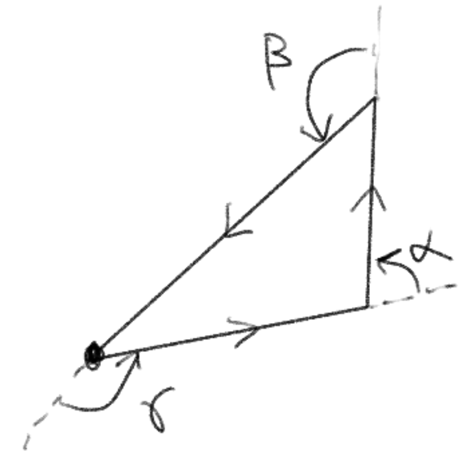


Surface in $\mathbb{R}^3$

study length, area, curvature

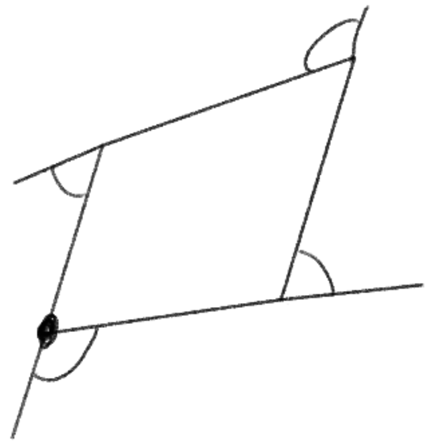




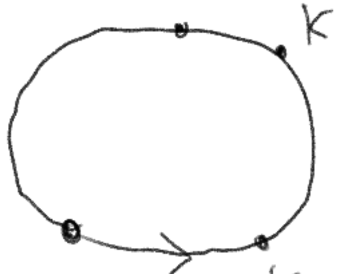


$\alpha+\beta+\gamma = 360^\circ$

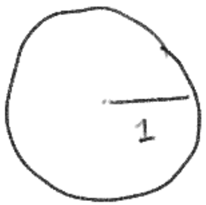
$= \underset{\sim}{2\pi}$



$\text{sum} = 360^\circ$



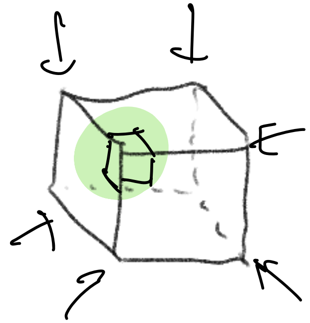
$\int K\,ds = \underset{\sim}{2\pi}$

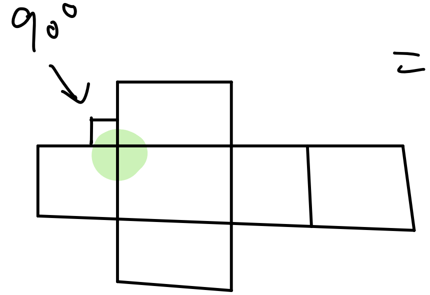


$S = 2\pi r$

$= \underset{\sim}{2\pi}$

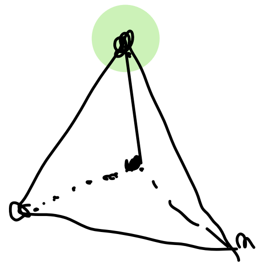
$\iint K\,dA = -4\pi$

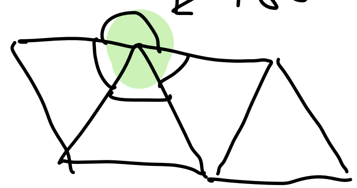




$90^\circ \times 8 = 720^\circ$

$= \underset{\sim}{4\pi}$



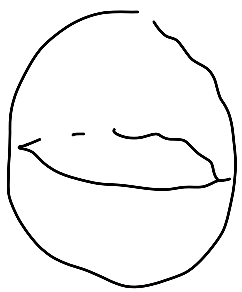


$180^\circ \times 4 = 720^\circ$

$= \underset{\sim}{4\pi}$

$\iint K\,dA = 4\pi$

$S = 4\pi r^2 = \underset{\sim}{4\pi}$



$r = 1$

1.1 Matrices

Definition 1.1.1 (Matrix). *Let m and n be positive integers. An $m \times n$* **matrix** *over $\mathbb{R}$ ($\mathbb{C}$) is a rectangular array of the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

m = number of rows

n = number of columns

where a_{ij}, $1 \le i \le m$, $1 \le j \le n$, are real (complex) numbers. We may also write a matrix as $A = [a_{ij}]$.

$$\text{2 rows} \left\{ \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}}_{\text{3 columns}} \right. = A$$

$a_{11} = 1$ $a_{12} = 2$ $a_{13} = 0$

$a_{21} = -1$ $a_{22} = 3$ $a_{23} = 4$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3×1

2×3

row columns

1. **Matrix addition**: *Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then*

$$[A+B]_{ij} = a_{ij} + b_{ij}$$

In other words

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{bmatrix}$$

2×3

2. **Scalar multiplication**: *Let $A = [a_{ij}]$ be a $m \times n$ matrix and c be a real (complex) number. Then*

$$[cA]_{ij} = ca_{ij}$$

In other words

$$c\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}.$$

$$(-2)\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -6 & -8 \\ -10 & -12 \end{bmatrix}$$

3×2

3. **Matrix multiplication**: *Let* $A = [a_{ij}]$ *be an* $m \times n$ *matrix and* $B = [b_{jk}]$ *be an* $n \times p$ *matrix. The matrix product of* A *and* B *is an* $m \times p$ *matrix and*

$$[AB]_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$$

for $1 \le i \le m$, $1 \le k \le p$. *Note that the ik-th entry of* AB *is the sum of the products of the corresponding entries in the i-th row of* A *and the k-th column of* B.

$$[AB]_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{4} & -2 & \boxed{3} \\ 1 & \boxed{-1} & 1 \end{bmatrix}$$

$2 \times 2 \leftarrow = \rightarrow 2 \times 3$ 　　　　 2×3

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{undefined} \qquad \frac{3}{0}$$

$2 \times 3 \neq 2 \times 2$

(Diagonal matrix). *An $n \times n$ matrix of the form*

$$D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

4×4 $\quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The $m \times n$ **zero matrix** *is the matrix which every entry equals to 0.*

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2×4 zero matrix

The **identity matrix** *of size n is the matrix*

$$I_n = I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A is $m \times n$

$I_m A = A$

$m \times m \quad m \times n \quad m \times n$

$A I_n = A$

$m \times n \quad n \times n$

0 is $m \times n$

$0 A = 0$

Remarks 1.1.4. *Let A, B, C be matrices.*

1. *AB is defined only when the number of columns of A is equal to the number of rows of B.*

2. *In general, $AB \neq BA$ even when they are both defined and of the same type.*

3. *In general, $AB = 0$ does not implies that $A = 0$ or $B = 0$.*

4. *In general, $AB = AC$ and $A \neq 0$ does not implies $B = C$.*

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}\underset{B}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}\underset{C}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Compare: $2x = 2y \overset{?}{\Rightarrow} x = y$

$\because$ "Inverse exists": $x = \frac{1}{2}\cdot 2x = \frac{1}{2}\cdot 2y = y$

1. $(AB)C = A(BC)$

2. $(A + B)C = AC + BC$ *and* $C(A + B) = CA + CB$

3. $c(AB) = (cA)B = A(cB)$

$$\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\2 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\3\\-1\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}4 & -2 & 3\\1 & -1 & 1\end{bmatrix}\begin{bmatrix}2\\3\\-1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\-2\end{bmatrix}$$

$$(A(BC))D = (AB)(CD)$$

Definition 1.1.6 (Transpose). *The* **transpose** *of an $m \times n$ matrix $A = [a_{ij}]$ is the $n \times m$ matrix A^T obtained by interchanging rows and columns of A, i.e.,*

$$[A^T]_{ji} = a_{ij}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$.

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \qquad A^T = \underset{3\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}}$$

Proposition 1.1.7 (Properties of transpose). *Let A and B be matrices.*

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = B^T A^T$

$$\underset{m\times n}{A}\ \underset{n\times p}{B} \quad m\times p \qquad \underset{p\times n}{B^T}\ \underset{n\times m}{A^T} \quad p\times m \qquad (AB)^T \neq A^T B^T$$

Definition 1.1.8 (Symmetric and anti-symmetric matrices). *Let A be an $n \times n$ matrix.*

1. *We say that A is a* **symmetric matrix** *if* $\boxed{A^T = A.}$
2. *We say that A is an* **anti-symmetric matrix** *(or a* **skew-symmetric matrix***) if $A^T = -A$.*

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix} \qquad A = A^T \quad \text{symmetric}$$

$$B = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \qquad B^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \qquad \text{anti-symmetric}$$

$$C = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \qquad E_{ij} = -E_{ji}$$

symmetric　　　anti-symmetric

$$\Rightarrow E_{ii} = -E_{ii}$$

non-trivial = non-zero

Proposition 1.1.11. *Suppose A is an $m \times n$ matrix where $m < n$. Then the homogeneous equation* $\boxed{A\mathbf{x} = \mathbf{0}}$ *has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$.*

$$\underset{m\times n}{A}\ \underset{n\times 1}{x} = \mathbf{0} = \underset{m\times 1}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} m\ 0\text{'s}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} \iff \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = 0 \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ is}$$

called trivial solution

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{bmatrix}$$

m = no of rows (equations)

n = no of columns (unknowns)

Definition 1.1.12 (Matrix inverse). *An $n \times n$ matrix A is said to be* **invertible**, *if there exists a matrix A^{-1} called the* **inverse** *of A such that*

$$AA^{-1} = A^{-1}A = I$$

$$2 \cdot \frac{1}{2} = 1$$

where I is the identity matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{if } ad-bc \neq 0$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition 1.1.13 (Properties of inverse). *Let A and B be two invertible $n \times n$ matrices over real (complex) numbers.*

$2^{-1} = \frac{1}{2}$ $\left(\frac{1}{2}\right)^{-1} = 2$

1. *The inverse A^{-1} is invertible and $(A^{-1})^{-1} = A$*

2. *For any nonnegative integer k, A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$. This allows us to define $A^{-k} = (A^{-1})^k$.*

3. *For any nonzero real (complex) number c, cA is invertible and $(cA)^{-1} = c^{-1}A^{-1}$*

4. *The product AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

5. *A^T is invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

$AB \ ? = I_n$

$$\begin{aligned}
&(AB)(B^{-1}A^{-1}) \\
&= A(\underbrace{BB^{-1}})A^{-1} \\
&= A I_n A^{-1} \\
&= A A^{-1} \\
&= I_n
\end{aligned}$$

1.2 Determinant

Definition 1.2.1 (Determinant). *Let n be a positive integer and*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$n \times n$

be an $n \times n$ matrix. The **determinant** *of A is denoted by*

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$A = (a_{11})$

$\det A = a_{11}$

and is defined inductively by

1. *For $n = 1$, we have $\det(A) = a_{11}$.*
2. *For $n > 1$, we have*

entries of A on 1st row

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n+1}a_{1n}\det(A_{1n})$$

where A_{ij}, $1 \le i, j \le n$ is the submatrix of A obtained by deleting the i-th row and the j-th column of A.

$n=2$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \det A = a_{11}\det(A_{11}) - a_{12}\det(A_{12})$$

$$= a_{11}\det(a_{22}) - a_{12}\det(a_{21})$$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad = a_{11}a_{22} - a_{12}a_{21} \qquad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

e.g. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4)-(2)(3) = -2 \qquad \diagdown - \diagup$

eg $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$

eg

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (1)\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - (2)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (3)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (1)\left((5)(9)-(6)(8)\right) - 2\left[(4)(9)-(6)(7)\right] + 3\left[32-35\right] = 0$$

or

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (1)(5)(9)+(2)(6)(7)+(3)(4)(8)$$

$$- (3)(5)(7) - (2)(4)(9) - (1)(6)(8) = 0$$

$$\diagdown\ \diagdown\ \diagdown\ -\ \diagup\ \diagup\ \diagup$$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{vmatrix} = \underbrace{0\begin{vmatrix} \quad \end{vmatrix}}_{0} - (1)\begin{vmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \underbrace{(0)\begin{vmatrix} \quad \end{vmatrix}}_{0} - \underbrace{(0)\begin{vmatrix} \quad \end{vmatrix}}_{0}$$

$$= 0 - (1)(2)(3)(4) + 0 - 0 = -24$$

Proposition 1.2.4 (Direct formula for determinant). *Let n be a positive integer and $A = [a_{ij}]$. Then*

$$\det(A) = \sum_{\sigma \in S_n} \underline{sign(\sigma) a_{1\overset{1}{\sigma(1)}} a_{2\overset{2}{\sigma(2)}} \cdots a_{n\overset{4}{\sigma(n)}}}$$

where S_n is the set of all permutations[1] *of $1, 2, \cdots, n$ and $sign(\sigma) = 1, -1$ when σ is a composition of even, odd number of transpositions*[2] *respectively.*

$\sigma \quad 1, 2, 3, 4$

$$\begin{vmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{vmatrix} = + a_{11} a_{22} a_{33} a_{34} - a_{12} a_{21} a_{33} a_{44}$$

$+ \ldots\ldots$

total 24 terms

Proposition 1.2.6 (Determinant under row and column operations). *Let A be an $n \times n$ matrix.*

1. *If B is obtained from A by multiplying a single row (or column) of A by a constant k, then* $\det(B) = k\det(A)$.

2. *If B is obtained from A by interchanging two rows (or columns) of A, then* $\det(B) = -\det(A)$.

3. *If B is obtained from A by adding a constant multiple of one row (or column) of A to another row (or column) of A, then* $\det(B) = \det(A)$.

$$\underbrace{\begin{vmatrix} 10 & 2 & 3 \\ 10 & 2 & 1 \\ 20 & 3 & 0 \end{vmatrix}}_{B=10A} = 10 \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 3 & 0 \end{vmatrix}}_{A} \overset{R_2+(-1)R_1}{=} 10 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 2 & 3 & 0 \end{vmatrix}$$

Upper triangular matrix

$$\overset{R_3+(-2)R_1}{=} 10 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -1 & -6 \end{vmatrix} = -10 \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -6 \\ 0 & 0 & -2 \end{vmatrix}$$

diagonal

zero

$= -10(1)(-1)(-2)$

$= 20$

Proposition 1.2.8 (Further properties of determinant). *Let A be an $n \times n$ matrix.*

1. *If A has a row (or column) consisting entirely of zeros, then* $\det(A) = 0$.

2. *If two rows (or columns) of A are identical, then* $\det(A) = 0$.

3. *If A is an upper triangular matrix, that is,*

$$A = \begin{pmatrix} a_{11} & & & * \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

   Then $\det(A) = a_{11}a_{22}\cdots a_{nn}$. *In particular, the determinant of a diagonal matrix is the product of its diagonal entries.*

4. $\det(cA) = c^n \det(A)$ *for any* $c \in \mathbb{R}$. *(Caution!* $\det(cA) \neq c\det(A)$*)*

5. $\det(A^T) = \det(A)$

det

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

4.

$$\underbrace{\begin{vmatrix} 2a & 2b \\ 2c & 2d \end{vmatrix}}_{2A} = 2\begin{vmatrix} a & b \\ 2c & 2d \end{vmatrix} = (2)^2 \underbrace{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}_{A}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

matrix

Write

$$A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are column vectors of A. We may consider $\det(A)$ as a real valued function of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. Then the determinant is a function from $(\mathbb{R}^n)^n = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ to $\mathbb{R}$ which is characterized by the following properties.

$$\begin{vmatrix} 2 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -3$$

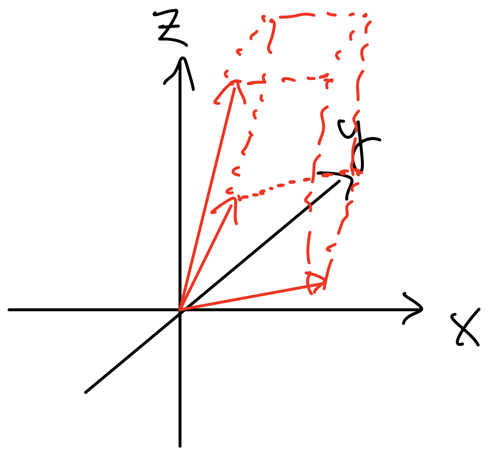
$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ a_1 & a_2 & a_3 \end{array}$$

$$\det : \{\text{Matrix}\} \to \mathbb{R}$$

or

$$\det : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$$

$$a_1, a_2, a_3 \in \mathbb{R}^3$$

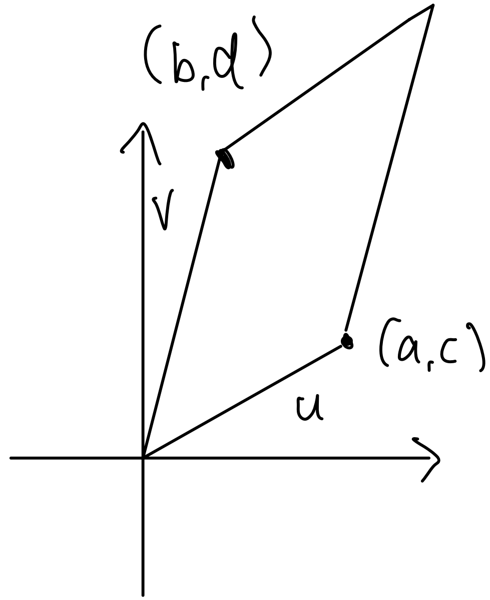


$$\det[a_1, a_2, a_3] = \underset{\uparrow \atop \text{orientation}}{\pm} \text{Volume}$$

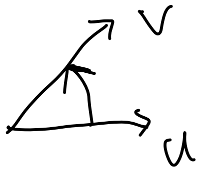
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm \text{ Area of } ▱$$

$u \quad v$

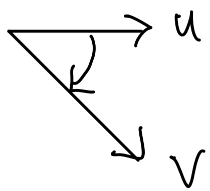
orientation



$+$



$-$



Theorem 1.2.5 (Characterizing properties of determinant). *The determinant* $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ *is a function characterized by the following properties.*

1. *(Multilinearity) For any* $k = 1, 2, \ldots, n$ *and* $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} &\det[\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}, \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n] \\ = \; &\alpha \det[\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}, \mathbf{u}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n] + \beta \det[\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}, \mathbf{v}, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n]. \end{aligned}$$

2. *(Anti-symmetry) For any* $1 \le i < j \le n$,

$$\det[\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_n] = -\det[\mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_n].$$

3. *(Determinant of identity) We have*

$n=2 \quad e_1 = (1,0)^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$e_2 = (0,1)^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\det[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = 1$$

where

$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$$\mathbf{e}_i = (0, \ldots, 0, 1, 0 \ldots, 0)^T \in \mathbb{R}^n$$

is the n *column vector with the* i*-th entry equals to* 1 *and all other entries equal to* 0. *In other words,* $\det(I) = 1$ *where* I *is the* $n \times n$ *identity matrix.*

$n=3 \quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

Furthermore, if $f : (\mathbb{R}^n)^n \to \mathbb{R}$ *is a function which is multilinear, anti-symmetric and satisfies*

$$f(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = k,$$

then

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = k \det(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$

for any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{R}^n$.

Proposition 1.2.9. *Let A and B be two $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

$$\begin{array}{cc} A & B \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \end{array}$$

$$= \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} \right]$$

A $n \times n$ $\quad v_i$ $n \times 1$

Av_i $n \times 1$

<u>Pf</u> $B = [v_1, v_2, \cdots, v_n]$ $\qquad v_i \in \mathbb{R}^n$

$AB = [Av_1 \ Av_2 \ \cdots \ Av_n]$

Define $f : (\mathbb{R}^n)^n \to \mathbb{R}$,

$f(v_1, v_2, \cdots, v_n) = \det([Av_1 \ Av_2 \ \cdots \ Av_n])$

Note f is multi-linear, anti-symmetric

<u>Thm</u> $\Rightarrow$ $f(v_1, \cdots, v_n) = f(e_1, e_2, \cdots, e_n) \det(v_1, v_2, \cdots, v_n)$

$\det(AB) = \det(A) \det B$

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the (i,j) **cofactor** by

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

M_{ij} is obtained from A by deleting i-th row, j-th column

adjugate matrix *of* A, *that is* $[\mathrm{adj}(A)]_{ij} = A_{ji}$

Proposition 1.2.11. *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix. Then* A *is invertible, that is, the inverse* A^{-1} *of* A *exists, if and only if* $\det(A) \neq 0$. *Moreover if* A *is invertible, then*

$$A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$$

where $\mathrm{adj}(A)$ *is the adjugate matrix of* A.

$AB = BA = I_n$

$B = A^{-1}$

$$\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{A}^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$M_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} \qquad M_{23} = \begin{bmatrix} a & b \\ g & h \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} = \frac{1}{\det A}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Definition 1.2.12 (Trace). *Let* $A = [a_{ij}]$ *be an* $n \times n$ *matrix. The* **trace** *of* A *is defined by*

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

$$\mathrm{tr}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9 = 15 \qquad \mathrm{tr}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1+4$$

Proposition 1.2.13 (Properties of trace). *Let* A, B *be* $n \times n$ *matrices and* $k \in \mathbb{R}$. *Then*

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(kA) = k\mathrm{tr}(A)$
3. $\boxed{\mathrm{tr}(AB) = \mathrm{tr}(BA)}$

$$(\det A)(\det B) = (\det B)(\det A)$$

$$\boxed{\det(AB) = \det(BA)}$$

1.3 Vectors (element of $\mathbb{R}^n$)

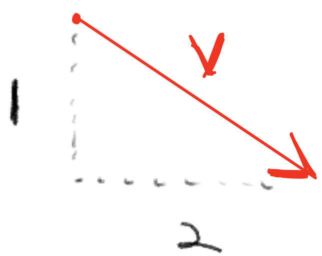
Definition 1.3.1 (Euclidean space). *Let n be a positive integer. The n dimensional* **Euclidean space** *is the set*

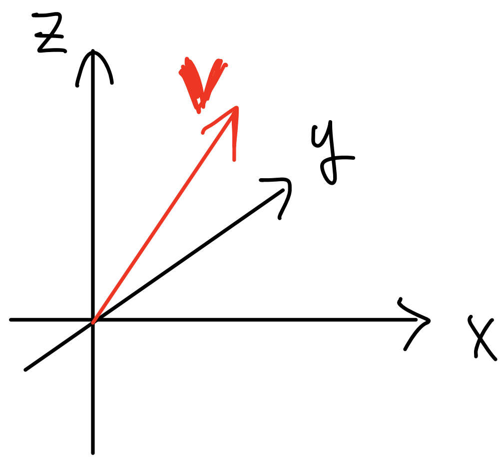
$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for any } i = 1, 2, \ldots, n\}.$$

Notation: $\mathbf{v}$, $\vec{v}$, $\bar{v}$, v

$v = (1, 2, 3)$

e.g. $v = (2, -1)$ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$





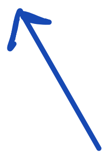
DSE: $v = 2i - j$

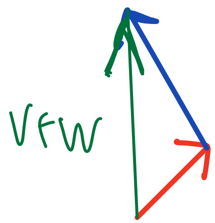
Definition 1.3.2 (Vector addition and scaler multiplication).

1. **Vector addition**: *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *Define*
$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

2. **Scalar multiplication**: *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Define*
$$\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \ldots, \alpha v_n).$$

eg. $\vec{v} = (1, 1)$ $\qquad$ $\vec{w} = (-1, 2)$



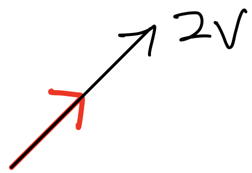


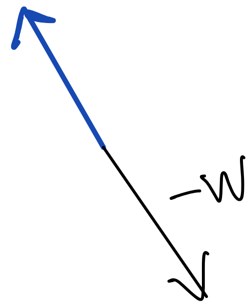
$\vec{v} + \vec{w} = (0, 3)$

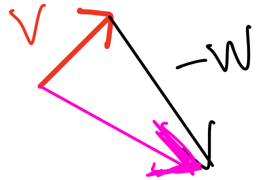
$2\vec{v} = (2, 2)$

$-\vec{w} = (1, -2)$

$\vec{v} - \vec{w} = (2, -1)$







Definition 1.3.3 (Scalar product). *Let* $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *The* **scalar product**, *or* **dot product**, *of* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined by*

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Note that the scalar product of two vectors is a number, not a vector.

$$\langle (1,2,3), (4,5,6) \rangle = (1)(4) + (2)(5) + (3)(6) = 32$$

Another notation $(1,2,3) \bullet (4,5,6)$

Proposition 1.3.4 (Properties of scalar product). *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $\alpha, \beta \in \mathbb{R}$. *Then*

1. *(Bilinear):*

$$= \langle \alpha u, w \rangle + \langle \beta v, w \rangle$$

$$\langle \alpha \mathbf{u} + \beta \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \beta \langle \mathbf{v}, \mathbf{w} \rangle$$

2. *(Symmetric):*

$$\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

3. *(Positive definite):*

$$\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$$

with equality holds if and only if $\mathbf{v} = \mathbf{0}$.

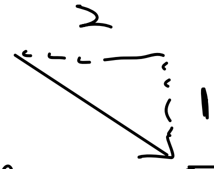
$$\langle V, V \rangle = V_1 V_1 + V_2 V_2 + \cdots + V_n V_n$$

$$= V_1^2 + V_2^2 + \cdots + V_n^2 \geq 0$$

Definition 1.3.6 (Norm). *Let* $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$. *The* **norm** *of* $\mathbf{v}$ *is defined as*

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

$V = (2, -1)$

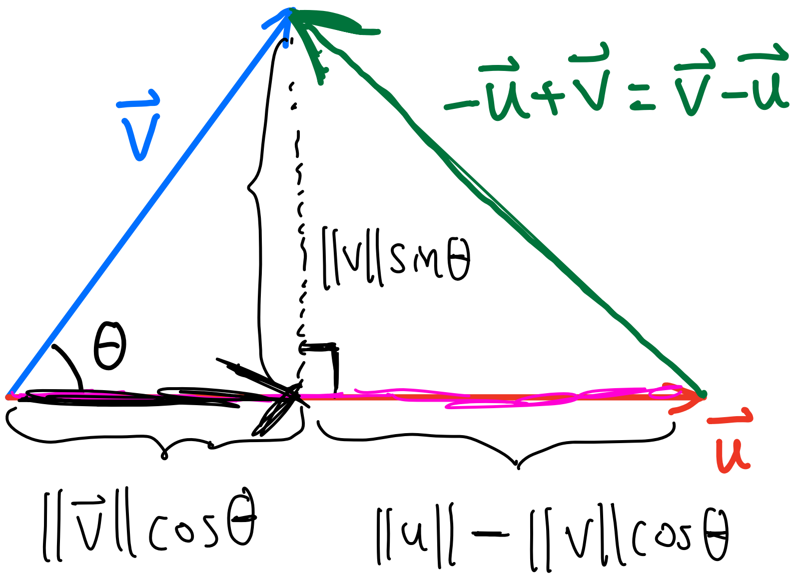


length $= \sqrt{2^2 + 1^2}$

Definition 1.3.7 (Unit vector). *We say that* $\mathbf{v} \in \mathbb{R}^n$ *is a* **unit vector** *if* $\|\mathbf{v}\| = 1$.

Prop $\langle u, v \rangle = \|u\| \, \|v\| \cos\theta$, where θ is angle between u, v ($\mathbb{R}^2$ or $\mathbb{R}^3$)

Pf



① $\|v - u\|^2 = (\|v\| \sin\theta)^2 + (\|u\| - \|v\| \cos\theta)^2$

$= \|v\|^2 \sin^2\theta + \|u\|^2 - 2\|u\|\|v\| \cos\theta + \|v\|^2 \cos^2\theta$

$= \|v\|^2 + \|u\|^2 - 2\|u\|\|v\| \cos\theta$

② $\|v - u\|^2 = \langle v - u, v - u \rangle$

$= \langle v, v \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle u, u \rangle$

$= \|v\|^2 + \|u\|^2 - 2\langle u, v \rangle$

Theorem 1.3.8 (Cauchy-Schwarz inequality). *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, *we have*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|$$

with equality holds if and only if $\mathbf{u} = \mathbf{0}$ *or* $\mathbf{v} = \alpha \mathbf{u}$ *for some real number* α.

$$-\|u\|\|v\| \leq \langle u, v \rangle \leq \|u\| \|v\|$$

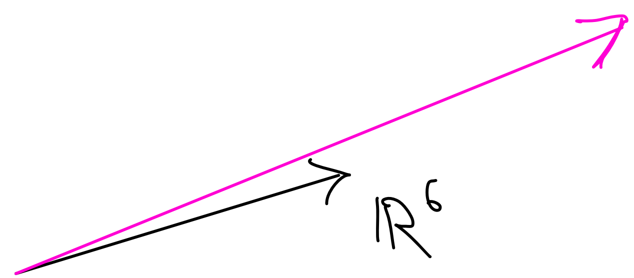
$$|x| \leq 3$$

$$-3 \leq x \leq 3$$

$$-1 \leq \frac{\langle u, v \rangle}{\|u\| \|v\|} \leq 1$$

Definition 1.3.10 (Angle between two vectors). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be two nonzero vectors. The* **angle** *between* $\mathbf{u}$ *and* $\mathbf{v}$ *is the unique* $\theta \in [0, \pi]$ *such that*

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|}$$



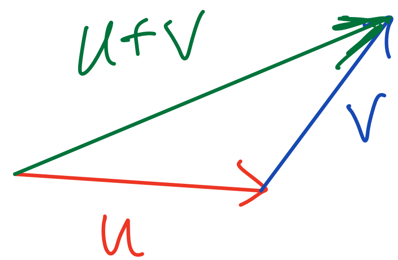
Proposition 1.3.9 (Properties of norm). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$.

1. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$
2. $\|\mathbf{v}\| \geq 0$ *with* $\|\mathbf{v}\| = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.
3. *(Triangle inequality):*
$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$
4. *(Parallelogram law):*
$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4}\left(\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2\right)$$

$$\|\alpha v\| = |\alpha| \|v\|$$

$$\|-3v\| = |-3| \|v\| = 3\|v\|$$

3



Pf 3
$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v \rangle \\
&= \langle u,u \rangle + 2\langle u,v \rangle + \langle v,v \rangle \\
&\leq \|u\|^2 + 2|\langle u,v \rangle| + \|v\|^2 \\
&\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

$$|\langle u,v \rangle| \leq \|u\| \|v\|$$

Definition 1.3.11 (Orthogonal vectors). *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *be two vectors. We say that* $\mathbf{u}$ *and* $\mathbf{v}$ *are* **orthogonal** *and write* $\mathbf{u} \perp \mathbf{v}$ *if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

perpendicular

$$\langle u, v \rangle = \|u\| \|v\| \cos\theta$$

$\boxed{12:35}$

Definition 1.3.12 (Cross product). *Let* $\mathbf{u} = (u_1, u_2, u_3)$,$\mathbf{v}= (v_1, v_2, v_3) \in \mathbb{R}^3$ *be two vectors in* $\mathbb{R}^3$. *The* **cross product**, *or* **vector product**, *of* $\mathbf{u}$ *and* $\mathbf{v}$ *is defined by*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

where $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$.

$u = (2,3,5) \quad v = (1,2,3)$

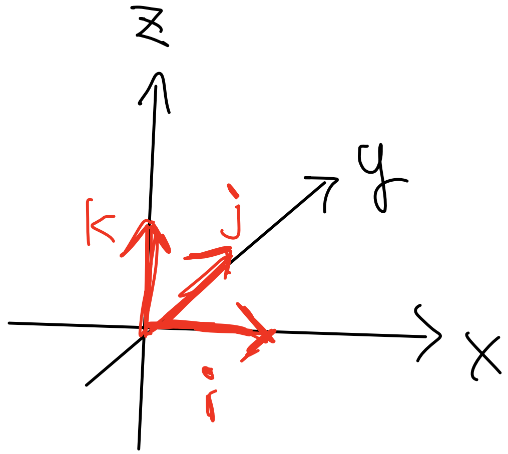
$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} i - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} k$$

$$= (-1)i - (1)j + (1)k$$

$$= (-1,0,0) - (0,1,0) + (0,0,1)$$

$$= (-1,-1,1)$$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$
$$= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$



Proposition 1.3.13 (Properties of cross product).

1. $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$

2. *(Bilinear) For any* $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,
$$(\alpha\mathbf{u} + \beta\mathbf{v})\times\mathbf{w} = \alpha\mathbf{u}\times\mathbf{w} + \beta\mathbf{v}\times\mathbf{w}$$

$$v \times u = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

3. *(Anti-symmetric) For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,
$$\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v}$$

4. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, *we have* $\mathbf{u}\times\mathbf{v} \perp \mathbf{u}$ *and* $\mathbf{u}\times\mathbf{v} \perp \mathbf{v}$, *that is*
$$\underbrace{\langle \mathbf{u}\times\mathbf{v}, \mathbf{u}\rangle}_{\begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}} = \langle \mathbf{u}\times\mathbf{v}, \mathbf{v}\rangle = 0$$

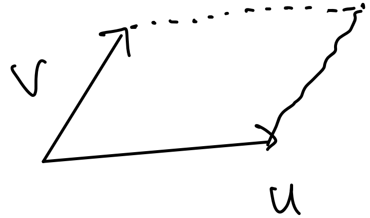
$u \times v \perp u$

$u \times v \perp v$

5. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta\, \mathbf{n}$$

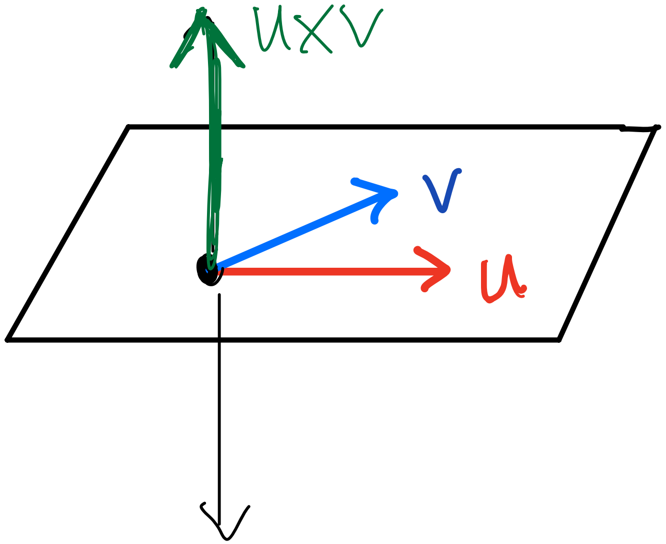
$\mathbb{R}^3$

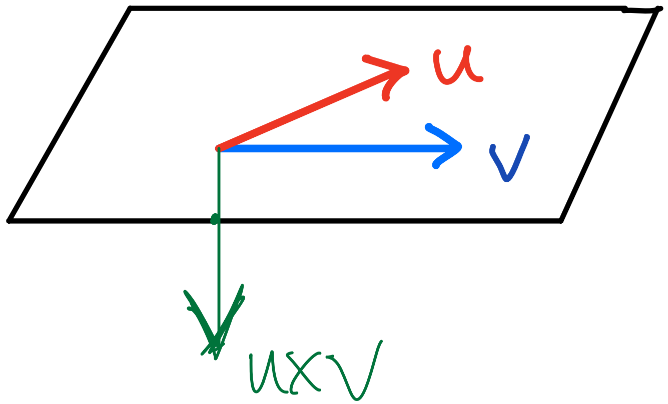


where θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$, *and* $\mathbf{n}$ *is the unit vector normal to the plane spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *with direction determined by the right hand rule. In other words,*

(a) $\|\mathbf{u} \times \mathbf{v}\|$ *is equal to the area of the parallelogram spanned by* $\mathbf{u}$, $\mathbf{v}$.

(b) $\mathbf{u} \times \mathbf{v}$ *is normal to the plane spanned by* $\mathbf{u}$ *and* $\mathbf{v}$ *with direction determined by the right hand rule.*





Definition 1.3.14 (Scalar triple product). *Let* $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. *The* **scalar triple product** *of* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is defined by*

$$\underbrace{\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle}_{\text{defn}} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

$$v \times w = \begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

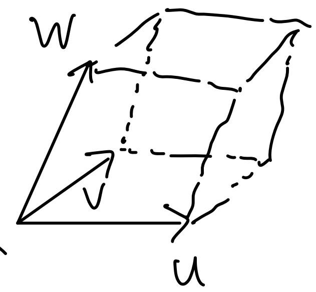
$$\text{L.H.S.} = \langle u, v \times w \rangle = \left\langle (u_1, u_2, u_3), \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \right\rangle$$

$$= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \text{R.H.S.}$$

$\det A = \det A^T$

Rmk

$\langle u, v \times w \rangle = \pm \operatorname{Vol} \Big($  $\Big)$

Proposition 1.3.16 (Cyclic property of scalar triple product)**.** *Let* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *be three vectors. We have*

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \times \mathbf{u} \rangle = \langle \mathbf{w}, \mathbf{u} \times \mathbf{v} \rangle$$

The following three identities are useful in studying curvature of surfaces.

Proposition 1.3.17.

1. *For any* $\mathbf{u}_1, \mathbf{v}_1, \mathbf{u}_2, \mathbf{v}_2 \in \mathbb{R}^3$,

$$\begin{vmatrix} \langle \mathbf{u}_1, \mathbf{u}_2 \rangle & \langle \mathbf{u}_1, \mathbf{v}_2 \rangle \\ \langle \mathbf{v}_1, \mathbf{u}_2 \rangle & \langle \mathbf{v}_1, \mathbf{v}_2 \rangle \end{vmatrix} = \langle \mathbf{u}_1 \times \mathbf{v}_1, \mathbf{u}_2 \times \mathbf{v}_2 \rangle$$

2. *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\begin{vmatrix} \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} = \|\mathbf{u} \times \mathbf{v}\|^2$$

3. *For any* $\mathbf{x}_{11}, \mathbf{x}_{12}, \mathbf{x}_{21}, \mathbf{x}_{22}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\begin{aligned} & \begin{vmatrix} \langle \mathbf{x}_{11}, \mathbf{u} \times \mathbf{v} \rangle & \langle \mathbf{x}_{12}, \mathbf{u} \times \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{u} \times \mathbf{v} \rangle & \langle \mathbf{x}_{22}, \mathbf{u} \times \mathbf{v} \rangle \end{vmatrix} \\ = & \begin{vmatrix} \langle \mathbf{x}_{11}, \mathbf{x}_{22} \rangle - \langle \mathbf{x}_{12}, \mathbf{x}_{21} \rangle & \langle \mathbf{x}_{11}, \mathbf{u} \rangle & \langle \mathbf{x}_{11}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{22}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{22}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} \\ & - \begin{vmatrix} 0 & \langle \mathbf{x}_{12}, \mathbf{u} \rangle & \langle \mathbf{x}_{12}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{u} \rangle & \langle \mathbf{u}, \mathbf{v} \rangle \\ \langle \mathbf{x}_{21}, \mathbf{v} \rangle & \langle \mathbf{v}, \mathbf{u} \rangle & \langle \mathbf{v}, \mathbf{v} \rangle \end{vmatrix} \end{aligned}$$

Vector-valued functions

real-valued function $f(t) = 2t+1$

Vector-valued function $v(t) = (t, t^2)$ output $\mathbb{R}^2$

$w(t) = (\cos t, \sin t, t)$

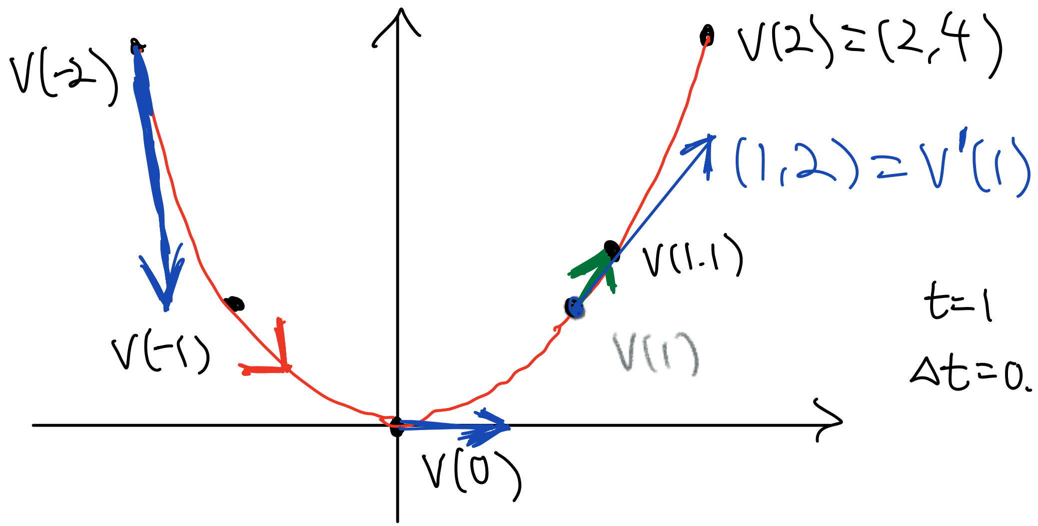
$0 \le t \le 2\pi$

$\begin{cases} x = t \\ y = t^2 \end{cases}$

$\Rightarrow y = x^2$

Derivative

$$v'(t) = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}$$



$t = 1$

$\Delta t = 0.1$

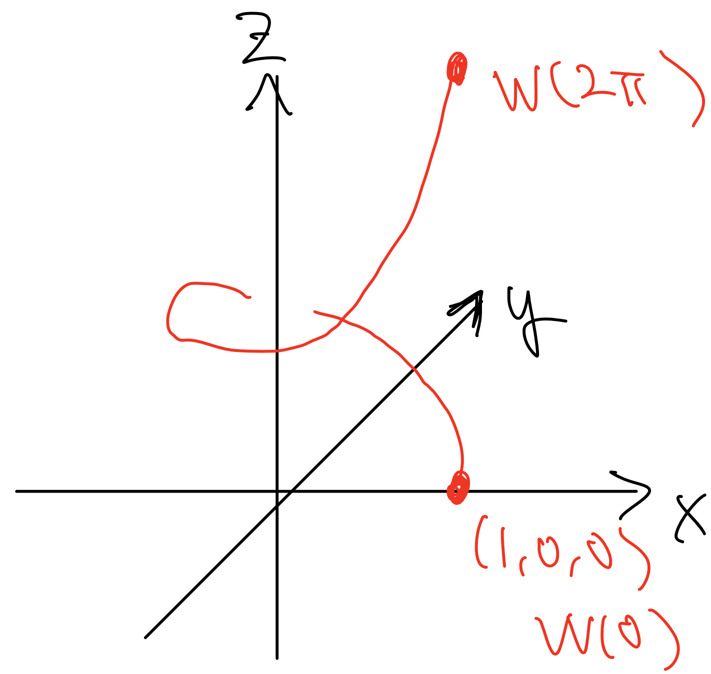
$v(t) = (x(t), y(t))$

$v'(t) = (x'(t), y'(t))$

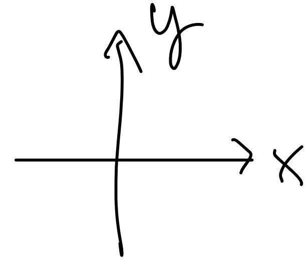
eg $v(t) = (t, t^2)$

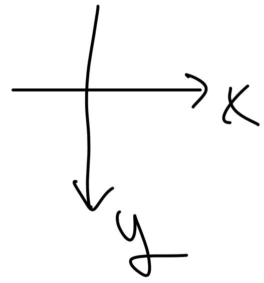
$v'(t) = (1, 2t)$

$v'(1) = (1, 2)$



Proposition 1.3.34 (Rules for derivative of vector valued functions). *Let* $\mathbf{u}(t), \mathbf{v}(t), \mathbf{w}(t)$ *be differentiable vector valued functions and* $\alpha(t)$ *be real valued function.*





1. $\dfrac{d}{dt}(\mathbf{u}+\mathbf{v}) = \dfrac{d\mathbf{u}}{dt} + \dfrac{d\mathbf{v}}{dt}$

2. $\dfrac{d}{dt}(\alpha\mathbf{v}) = \alpha\dfrac{d\mathbf{v}}{dt} + \dfrac{d\alpha}{dt}\mathbf{v}$

3. $\dfrac{d}{dt}\langle\mathbf{u},\mathbf{v}\rangle = \langle\dfrac{d\mathbf{u}}{dt},\mathbf{v}\rangle + \langle\mathbf{u},\dfrac{d\mathbf{v}}{dt}\rangle$

4. $\dfrac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \dfrac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\dfrac{d\mathbf{v}}{dt}$

5. $\dfrac{d}{dt}\langle\mathbf{u},\mathbf{v}\times\mathbf{w}\rangle = \langle\dfrac{d\mathbf{u}}{dt},\mathbf{v}\times\mathbf{w}\rangle + \langle\mathbf{u},\dfrac{d\mathbf{v}}{dt}\times\mathbf{w}\rangle + \langle\mathbf{u},\mathbf{v}\times\dfrac{d\mathbf{w}}{dt}\rangle$

$$\frac{d}{dt}\left[t(\cos t, \sin t)\right] = \frac{dt}{dt}(\cos t, \sin t) + t\frac{d}{dt}(\cos t, \sin t)$$

$$= (\cos t, \sin t) + t(-\sin t, \cos t)$$

eg $\dfrac{d}{dt}\left((1,t,t^2)\times(t,2t,3t)\right)$

$$= \left(\frac{d}{dt}(1,t,t^2)\right)\times(t,2t,3t) + (1,t,t^2)\times\left(\frac{d}{dt}(t,2t,3t)\right)$$

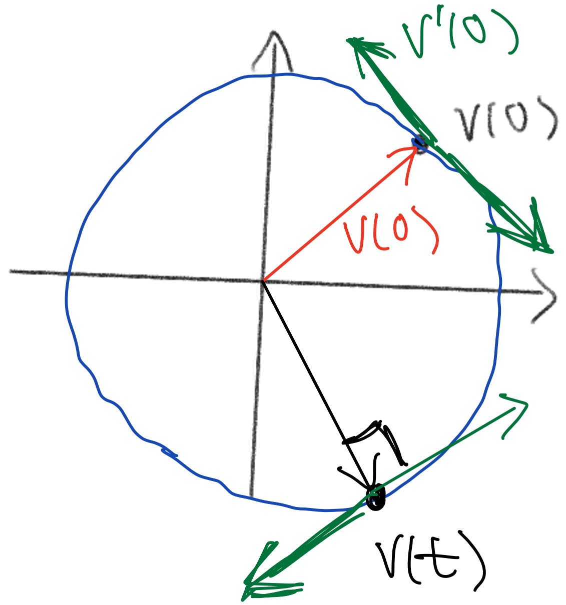
Lemma 1.3.35. *Let* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *be two vector valued functions.*

1. *If* $\langle \mathbf{u}(t), \mathbf{v}(t)\rangle$ *is constant, then for any* t*, we have*

$$\langle \mathbf{u}'(t), \mathbf{v}(t)\rangle = -\langle \mathbf{u}(t), \mathbf{v}'(t)\rangle.$$

2. *If* $\|\mathbf{v}(t)\|$ *is constant, then for any* t*, we have*

$$\langle \mathbf{v}'(t), \mathbf{v}(t)\rangle = 0.$$



Pf ① $\langle u(t), v(t)\rangle = c$ constant

$\frac{d}{dt}$ ↓

$\langle u'(t), v(t)\rangle + \langle u(t), v'(t)\rangle = 0$

② $\|v(t)\|^2 = \langle v(t), v(t)\rangle$ is const

By ① $\langle v'(t), v(t)\rangle = -\langle v(t), v'(t)\rangle$

$\Rightarrow \langle v'(t), v(t)\rangle = 0$